The Emphasis on Applied Mathematics Today and Its Implications for the Mathematics Curriculum

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1. Across the country, and beyond the borders of the United States, the cry is being heard that we mathematicians should be concerning ourselves more, both in our research and in our teaching, with applications of mathematics. It is being argued that we have been overemphasizing mathematics itself, the autonomous discipline of mathematics, at the expense of due attention to its usefulness, to its role in science, in engineering, in the conduct of modern society. Some put it crudely—there is too much "pure mathematics," too little "applied mathematics."

The argument is an important one; it is rendered the more crucial by its relation to a critical problem now confronting the profession of academic mathematicians—the declining enrollment in the mathematics major and, in particular, in the more traditional upper division courses. How can this process be arrested and reversed? The answer is seen to be closely connected with the idea that we should somehow seek to make our mathematics courses more relevant to the needs and interests of today's students, without any sacrifice of standards or of integrity.

Many groups of mathematicians and mathematics educators have devoted considerable effort to coming to grips with these related problems. There is the report† of the NRC Committee on Applied Mathematics Training; a panel of CUPM, under the chairmanship of Professor Alan Tucker, is in the process of producing sample curricula with a decidedly "applied" flavor; there is a joint MAA–SIAM Committee considering undergraduate and graduate courses; there are the recommendations of the PRIME 80 Conference; and much else.

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† The Role of Applications in the Undergraduate Mathematics Curriculum, AMPS, National Research Council, Washington (1979).

At this conference we have listened to some of the best applied mathematicians describing recent work in their fields. With these stimulating talks fresh in our minds it is natural to ask ourselves, as teachers of mathematics, two questions. First, what emerges from these talks as the distinctive quality of good applied mathematics; and, second, how can we prepare our students to do the kind of work these mathematicians do and which they have so vividly described.

First it is plain, above all else, from these talks that to do good work in applied mathematics one needs to be a good mathematician. Of course, one needs more, but one certainly needs this. Moreover, there is no natural division of mathematics itself into applicable mathematics and mathematics *sui generis*—our nine speakers have used, in their talks, apart from the obvious areas of ordinary and partial differential equations, material from combinatorics, commutative algebra, the theory of jets, algebraic geometry, Lie groups and Lie algebras, differential topology, algebraic topology, fibre bundle theory, deformation of complex structures, singularity theory and functional analysis.* Moreover, the talks have been distinguished from talks in so-called pure mathematics not by the manner of treatment, or the rigor of the argument, but by the "real world" motivation for the mathematical problem. Thus if our students are to be able to follow in these footsteps, they must have a broad and deep education in mathematics and an attitude of a very positive kind towards applications. This conclusion is in striking agreement with that of the NRC Committee referred to above, where both points are made with great emphasis, and where, in keeping with the conclusion that there is a fundamental unity encompassing the whole of mathematics, pure and applied, a plea is made for a broad-based major in the mathematical sciences, giving *all* students the encouragement and opportunity to acquire familiarity with the way mathematics is, in fact, applied.

Can we then make a reasonable working definition of applied mathematics? Let me attempt one, based on a definition of applied analysis given by Kaper and Varga.† We propose the following:

> "The term *applied mathematics* refers to a collection of activities directed towards the formulation of mathematical models, the analysis of mathematical relations occurring in these models, and the interpretation of the analytical results in the framework of their intended application. The objective of an applied mathematics research activity is to obtain qualitative and quantitative information about exact or approximate solutions. The methods used are adapted from all areas of mathematics; because of the universality of mathematics, one analysis often leads, simultaneously, to applications in several diverse fields."

* It is true that the talk by Professor Oster is different in kind from the others. It was the deliberate plan of the program committee to put into the program one such exceptional talk. The plan was splendidly vindicated!

† Hans G. Kaper and Richard S. Varga, Program Directions for Applied Analysis, Applied Mathematical Sciences Division, Department of Energy (1980). I am most grateful for the opportunity to see this paper, from which I have drawn many ideas.

In this paper we will first make some general remarks about the nature of mathematical modeling—these will form the content of the next section. We will then suggest, in the third section, that the basic method of applied mathematics is not, in fact, as distinctive as it at first appears; that, in fact, it has much in common with processes of abstraction and generalization that go on within mathematics itself. These remarks will lead us to the conclusion that, by modifying our approach to the curriculum in certain ways that give expression to the unity of the mathematical sciences which it is our task to present to our students, we may prepare them to become mathematicians, able and willing to place their knowledge and talents at the service of problems coming from within or without mathematics. Thus might the sterile antagonism which one sometimes finds today between pure and applied mathematics—and pure and applied mathematicians—be eliminated by abandoning these labels and reverting to the notion of a single indivisible discipline, mathematics.

2. What can be said in general of the process of mathematical modeling? The following schema seems to reflect the methods described by the illustrious—and successful—mathematicians who have spoken at this conference:

$$\boxed{\text{Real-life problem}} \xrightarrow{①} \boxed{\text{Scientific model}} \xrightarrow{②} \boxed{\text{Mathematical model}}$$

Step ① occurs whenever a mature science is involved; it may well happen in the soft social sciences that one proceeds directly from real-life problem to mathematical model. Such a process is dangerous, because it is within the scientific model that one locates the measurable constructs about which one theorizes, quantitatively and qualitatively, within the mathematical model. Thus the direct passage from problem to mathematical model, while often intellectually exciting, is open to the objection that one may well be using sophisticated mathematical tools to reason about extremely vague concepts involving very unreliable measurements.

Let us then assume that a scientific model is, indeed, articulated. This will consist, typically, of *objects* (observables, constructs) and *laws* (physical, chemical, biological) about the behavior of matter in the form of liquids, gases and solid particles. The selection of the appropriate scientific model, step ①, may be called *constructive analysis*; for example, in the study of energy systems (fission reactors, combustion chambers, coal gasification plants), the laws express the rate of flow of mass, momentum and energy between the components of the system.

Step ② consists of choosing a mathematical model for the analysis of the scientific system. The mathematical model is, by its very nature, both more abstract and more general than the scientific system being modeled. Thus the conservation laws for an incompressible viscous fluid lead to the Navier–Stokes equation, which is an *evolution* equation; but we may also derive *equi-*

librium equations leading to the study of bifurcation phenomena. Again, the study of non-linear wave mechanics may lead to the Korteweg–de Vries (KdV) equation for the behavior of long water waves in a rectangular channel.* Here the theory predicts solitary waves which interact but emerge unchanged—these are the solitons of modern theoretical physics. In these examples, the original constructive analysis leads to the next stage of mathematical analysis (in these examples, qualitative analysis) within the mathematical models. Typically, again, this stage consists of proofs of existence of solutions, together with a study of their uniqueness, stability (sensitivity to changes of parameter), and behavior over large time intervals (asymptotic analysis).

Quantitative analysis, however, also plays a key role in today's applied mathematics, due largely to the general availability of the high-speed computer. Numerical methods of a sophisticated kind have been developed; asymptotic and perturbation methods are widely used. One particularly important new tool of quantitative analysis that one may mention here is the "finite element" method, invented by Richard Courant, but rediscovered by engineers who saw its potential in conjunction with the computer. The method itself is, of course, undergoing improvement and refinement.

In several of the talks presented at this conference, we have seen evidence that our schema is often incomplete in one significant respect—the process of elaborating a mathematical model may well be *iterated.* Thus our scientific model may lead to a first-order differential equation which can be interpreted as a dynamical system and thus embedded in the theory of vector bundles or, more generally, fibre bundles. Thus the existence of a solution is translated, first, into an integrability problem and then into a cross-section problem to which we may apply the techniques of obstruction theory. Thus it would be very misleading to think of the process of abstraction and generalization as a one-stage procedure; by the same token it is a mistake to think of an area of mathematics as ineffably "pure" because all its direct, immediate contacts are with other areas of mathematics. This simplistic view would have found no favor with our speakers at this conference, whose views on the applicability of mathematics and on the relations between so-called pure and applied mathematics excluded any possibility of a rigid distinction being made.

Another very striking feature emerged from a consideration of the contributions to this conference. The mathematical content of the talks bore strong testimony to that universality of mathematics to which attention is drawn in the description of the nature of applied mathematics quoted above. For mathematicians adopting very different starting points in their investigations—control systems, the study of porous media, embryogenesis, bifurcation theory and turbulence—found themselves concerned with

* More precisely, the KdV equation describes the propagation of waves of small amplitude in a dispersive medium.

significantly overlapping domains of mathematics in the design and analysis of their models. The Navier–Stokes equation, for example, together with its linearization, figured in many contributions; and questions of stability, naturally, arose frequently when systems of partial differential equations were involved. Thus it may fairly be said that the talks displayed the salient features of good mathematics in action—its unity, its subtlety, its diversity, its power and its universality.

It is plain that if our students are to be able to apply mathematics effectively, they must gain some understanding and experience of the art of mathematical modeling. We do not recommend a special modeling course; rather, the modeling process should be explicitly discussed when an application of mathematics to a scientific problem is in question. We would ourselves recommend that the discussion include the following component items: the selection of a suitable problem; the development of an appropriate model; the collection of data; reasoning within the model (qualitative analysis); calculations (quantitative analysis); reference back to the original problem to test the validity of a solution; modification of the model; generalization of the model as a conceptual device. Moreover, the entire modeling process must be set against the contemporary background of a strong computer capability (actual or assumed).

However, it will be our claim in the next section that it is unnecessary to separate applications from the rest of mathematical activity in order to emphasize the processes named above. We will suggest, in fact, that good "applied" mathematics and good "pure" mathematics have a great deal in common, and that this complementarity should be reflected in the undergraduate curriculum.

3. The question I wish to consider in this section is this—how special to applied mathematics are the techniques and procedures described in the previous section? Of course, in one sense they certainly are special; for if we start off with a "real world" problem and apply mathematical reasoning, we are *ipso facto* doing applied mathematics. Thus for the question to make any sense, we must allow that the original problem to be tackled mathematically could itself be a mathematical problem. We would then claim that the process of abstraction which is characteristic of the schema we described in Section 2 also features in work within mathematics itself. Let us immediately give an example; this is admittedly a relatively trivial piece of mathematics, but it is without doubt a piece of mathematics currently pertaining to the undergraduate curriculum. We allow ourselves, here and subsequently, to consider the closely related processes of abstraction and generalization.

Suppose that it is observed that $5^6 \equiv 1 \bmod 7$. This may be verified empirically—thus we compute $5^6 - 1$, obtaining 15624 and check that 15624 is exactly divisible by 7. This argument is compelling and convincing—but unsatisfactory. We do not feel, with this demonstration, that we understand *why* the assertion is true.* The situation is ripe for generalization; we make a mathematical model! We conjecture, and then prove, that if p is any prime number and if a is prime to p, then (Fermat's Theorem) $a^{p-1} \equiv 1 \bmod p$. Notice that this is a generalization, not an abstraction, because we are still talking about rational integers. However, we may not feel entirely satisfied with the generality of Fermat's Theorem. We could proceed in one direction to Euler's Theorem; or we could regard Fermat's Theorem as itself a special case of Lagrange's Theorem that the order of a subgroup of a finite group divides the order of the group. This latter development involves abstraction as well as generalization, for we are now discussing abstract groups, and postulate, in our abstract system, only one binary operation (whereas the integers admit two, and both were involved in our original demonstration that $5^6 \equiv 1 \bmod 7$), which need not even be commutative.

A particular feature of this example is the iteration of the modeling (generalizing) process; as we saw, this is also, frequently, a feature of applied mathematics. On the other hand, let us immediately admit that there is also a difference between modeling a non-mathematical problem, and the modeling we did here. In our example, we obtained incontrovertible proofs of our original congruence, and, of course, of related congruences, whereas, in applied mathematics, the mathematical reasoning can at best establish a scientific assertion as a good working hypothesis, a good approximation to the truth. Let us, however, give a second "mathematical" example to show that this difference is not so absolute.

There is a beautiful numerical process, based on our base 10 enumeration system, called *casting out* 9's. What is involved here, in mathematical terms, is the canonical ring projection $\theta\colon \mathbb{Z} \to \mathbb{Z}/9$; we use the residue ring $\mathbb{Z}/9$ because it is particularly easy to compute† θ. Now we may say that θ provides a means of modeling identities in $\mathbb{Z}$ by means of identities in $\mathbb{Z}/9$. This is a good checking procedure, because it is far easier to do calculations in $\mathbb{Z}/9$ than in $\mathbb{Z}$. However, we cannot *prove* identities in $\mathbb{Z}$ by modeling them by true identities in $\mathbb{Z}/9$; we can only *disprove* them by modeling them by false identities. We are here involved in the important mathematical process of *simplification* (with preservation of structure); there is a strong analogy here with the simplification involved in modeling a real-world situation.

If we are to do justice to abstraction, generalization and simplification as key processes within mathematics itself, we find ourselves led inevitably to give prominence to the essential unity of mathematics. In practical terms this means we must insist far less on the autonomy and (apparent) independence of the various mathematical disciplines and emphasize their (real) inter-

* We are thus in the unfortunate situation so typical of our students! They are compelled to accept but do not truly understand.

† In fact, θ extends, uniquely, to a ring homomorphism $\mathbb{Z}_3 \to \mathbb{Z}/9$, where $\mathbb{Z}_3$ is the ring of integers localized at the prime 3. Likewise the projection $\mathbb{Z} \to \mathbb{Z}/11$ is easily computable and extends to $\mathbb{Z}_{11} \to \mathbb{Z}/11$.

dependence. This poses severe problems in the design of curricula, but we believe that perhaps the most important desideratum is the breadth of view of the instructor.

Examples of interaction between different mathematical disciplines abound, at the undergraduate as at any other level. Thus we use topology in the foundations of real analysis (a continuous function from a compact metric space to a Hausdorff space is uniformly continuous); we use algebra to do topology (the fundamental group); we use complex variable theory to do algebra (the fundamental theorem of algebra); we use algebra to do geometry (the syzygy method for proving Desargues' Theorem in the coordinatized plane); we use linear algebra to study systems of linear differential equations (eigenvalues and eigenvectors). These and other examples can be presented as modeling one mathematical situation by means of another. It is our contention that this "applied perspective" should be adopted in mathematics itself—and not merely for the worthy reason that this will help the student to become familiar with the ways of doing applied mathematics!*

Are there essential differences between the methodologies of "pure" and "applied" mathematics? This is, in my view, a very interesting subject for research, with strong implications for the teaching of mathematics. My own thinking is still at a fairly primitive stage on this question, but let me offer one fairly obvious example of an essential difference.

* The place of geometry in the curriculum is, today, a special concern and a special problem. Students are arriving at our universities and colleges woefully ignorant of geometry and seriously lacking in any geometric intuition. These failings undoubtedly contribute to the difficulties they experience with the regular calculus sequence. Among the upper division courses we also find that courses in geometry are under-subscribed (along with certain other "traditional" offerings). Indeed it may happen that the only viable geometry course is a course designed for future high school teachers—this is viable in the sense that it will have an adequate enrollment, but it usually fails to do justice to geometry as a living branch of mathematics.

We would recommend that the geometric point of view figure prominently in virtually all undergraduate mathematics courses. This point of view allows one to conceptualize more easily; and geometry is a wonderful source of ideas and questions. Geometry, in this informal sense, may be thought of as partaking of the quality of both pure and applied mathematics—it is, after all, of all the branches of mathematics, that which is closest to the world of our experience.

It is probably not realistic to recommend an attempt to revive the study of geometry for its own sake, in courses devoted exclusively to the discipline. But geometry is very "real" to the students; it provides questions to which the disciplines of analysis and algebra provide answers. Without geometry, these latter disciplines must often seem to the students to answer questions they could never imagine themselves asking!

I would like to join Bert Kostant in making a special plea for a regular course in the curriculum on Lie groups. Here geometry, algebra and analysis come together in a theory of great power and importance to both pure and applied mathematics; it is, moreover, a subject rich in history.

Naturally, at the graduate level, we should find a great interest among faculty and students in algebraic and differential geometry, in view of the very significant advances currently being made in these subjects, as autonomous disciplines, in their relations to other parts of mathematics, and as suitable models for problems coming from engineering and the physical sciences.

Suppose we are modeling some physical phenomenon and produce a differential equation with certain boundary conditions. Suppose further that we prove that this mathematical system has no solution. The effect of this discovery is to discredit the model—we must have over-simplified (say, by linearizing) or we must have neglected some aspect of the physical situation which was, in fact, highly relevant to the dynamic process we were modeling. However, if we model a mathematical situation, the connection between the situation and the model is far closer. There is still, as in applied problems, the very difficult art of choosing a good model (e.g., a useful generalization); but if the problem in the model has no solution, then the original problem had no solution, either. This remains true whether we are generalizing or simplifying in constructing our model.

4. We would like to close this essay with a few remarks on problem-solving as a curricular or pedagogic device. The clamor for applications finds its echo at the pre-college and even undergraduate level in a strong plea (endorsed by the PRIME 80 Conference and the National Council of Teachers of Mathematics, as a key element in their platforms) for greater emphasis on problem-solving.

Now it is not in question—and should always have been obvious—that the principal reason for learning mathematics is that it enables one to solve problems. If certain programs have appeared to neglect this proposition then they are undoubtedly, to that extent, seriously defective. However, what is emerging from all the propaganda for problem-solving tends to be something very different in nature from a simple forceful recommendation to keep in mind why we learn mathematics. For the advocates of problem-solving seem to be arguing that we should be teaching problem-solving as an *alternative* to our traditional approach. Good pedagogical strategy should be "problem-oriented," they argue; and, if problem-solving is effectively taught, we need not trouble the students so much to absorb the "theory" which has hitherto proved a stumbling-block to them. An example of this attitude is to be found in the publishers' puff for a (very good) book on applied combinatorics which reads, "Its applied approach gives your students the *emphasis on problem-solving* that they need to participate in today's new fields. Rather than focus on theory, this text contains hundreds of worked examples with discussion of common problem-solving errors . . ."

Not for the first time, I must insist that this false dichotomy between the building and analysis of mathematical structure on the one hand and problem-solving on the other is dangerous. If problems are to be solved mathematically, the mathematical model must be chosen. Either it must already be available to the would-be problem-solver or he (or she) must be capable of developing it by the modification of existing mathematical models. Thus the investigator absolutely needs to know, to understand, and to be able to discriminate between different mathematical structures.

There are, it is true, certain problem-solving strategies and precepts which it is worthwhile enunciating explicitly. But these cannot serve as a substitute for a knowledge of mathematics. One of the most impressive features of the talks at this conference was the evidence of the vast array of mathematical knowledge at the disposal of the speakers, and of how much they actually needed for the particular investigations they described. It would be truly calamitous if the belief were to be spread abroad that difficult quantitative problems could be solved merely by becoming proficient in the field of problem-solving, allying a knowledge of a few general principles to "sound common sense." This is just a sophisticated restatement of the old egalitarian fallacy.

There is a further reason, too, less obvious at first, why it would be a mistake to concentrate so exclusively on problem-solving. For it is implicit in the concept of problem-solving that the problem has already been formulated; it is further suggested that it is a matter then of selecting the best mathematical model and successfully exploiting it. Thus the question of *how* we formulate good questions is totally ignored—and this is an essential question in scientific work. Moreover, the problem-solving approach ignores the fact that it is often the mathematical concept and the mathematical result which suggest the promising question. Frequently it is an advance in mathematics which enables us to see the true nature of a scientific problem more clearly and to pose the significant questions (this is eminently true, for example, of electromagnetic theory and, more recently, the theory of solitons). Thus the essential two-way flow between mathematics and science is lost in an exclusively problem-solving mode of instruction.*

None of this, of course, is to gainsay the stimulus which the attempt to solve interesting problems provides for the understanding and doing of mathematics. The UMAP modules can be extremely valuable as a component of a rich mathematical education. It should not be thought, however—and here I believe I have understood the intentions of the UMAP editorial board—that these modules are to be concentrated on applied topics. It is easy to supply a list of mathematical topics suitable for modular treatment; a brief sample might include maximum and minimum problems treated by various methods; thought-provoking paradoxes; linearization and linearity in mathematics; computational complexity; the geometry of 3-dimensional polyhedra; topics in combinatorics; algebraic curves; the classical groups. However, as these topics should suggest, the modules do not replace the systematic study of mathematics—they stimulate, enrich and enliven it. Ultimately we can only serve the purpose of mathematics education, in all its facets, by inculcating both the ability and the will to do mathematics and to use it. If, as some enthusiasts for a more applied curriculum and for more problem-solving rightly claim, the ability without the will leads to sterility, it is also true that the will without the ability leads to frustration. We avoid both these unpleasant consequences by teaching all of mathematics as a unity, emphasizing its unique generality and its immense power.

* Problem-solving may be characterized as "going from question to answer;" but scientific and mathematical progress often consists of going from answer to question.