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A geometric application of Runge's theorem

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Resum *(CAT)*
En aquest article donem una demostració simple de l'existència d'aplicacions harmòniques de qualsevol superfície de Riemann cap al pla complex $\mathbb{C} \equiv \mathbb{R}^2$. La nostra eina principal és la teoria d'aproximació per funcions holomorfes en superfícies de Riemann.

Abstract *(ENG)*
In this article we give a simple proof of the existence of proper harmonic maps from any open Riemann surface into the complex plane $\mathbb{C} \equiv \mathbb{R}^2$. Our main tool will be the Approximation Theory by holomorphic functions on Riemann surfaces.

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1. Introduction

The Runge and Mergelyan Theorems are the main results of Approximation Theory in one complex variable. The former, proved in 1885, asserts that every holomorphic function defined on an open neighbourhood of a compact set K of $\mathbb{C}$ can be uniformly approximated on K by entire functions, provided that the complement of K in $\mathbb{C}$ has no relatively compact connected components, see [11]. In the same line, Mergelyan [10] proved in 1951 that a continuous function $K \to \mathbb{C}$, which is holomorphic on K°, can be uniformly approximated on K by holomorphic functions on an open neighbourhood of K. Later, Bishop [5] extended these results to the context of open Riemann surfaces.

Theorem 1.1 (Runge–Mergelyan Theorem)**.** *Let $\mathcal{R}$ be an open Riemann surface and let $K \subset \mathcal{R}$ be a compact subset such that $\mathcal{R} \setminus K$ has no relatively compact connected components in $\mathcal{R}$. For any continuous function $f\colon K \to \mathbb{C}$ which is holomorphic on K° and any $\epsilon > 0$, there exists a holomorphic function $F\colon \mathcal{R} \to \mathbb{C}$ such that $\|F(p) - f(p)\| < \epsilon$ for all $p \in K$.*

The Runge and Mergelyan Theorems are useful in many different areas, e.g., complex analysis or surface theory. In particular, these tools have been exploited in the construction of minimal surfaces in the three-dimensional euclidean space $\mathbb{R}^3$. Recall that this class of surfaces is closely related to complex analysis through the Enneper–Weierstrass representation.

A fundamental problem in minimal surface theory is to understand how the conformal type (i.e., the type of the underlying Riemann surface) influences the global geometry of minimal surfaces. From an analytical point of view, an open Riemann surface is hyperbolic if and only if it admits negative non-constant subharmonic functions, and it is parabolic otherwise. This classification can also be explained in terms of Brownian motion of a particle over the surface; parabolicity is equivalent to the property that the Brownian motion visits any open set at arbitrarily large moments of time with probability 1. See the book of Grigor'yan [8] for more details.

Up to biholomorphisms, the only simply connected open Riemann surfaces are the unit disk $\mathbb{D}$ (of hyperbolic type) and the complex plane $\mathbb{C}$ (of parabolic type). Heinz [9] proved in 1952 that there do not exist harmonic diffeomorphisms between $\mathbb{D}$ and $\mathbb{C}$ with the euclidean metrics, extending the classical theorems by Riemann and Liouville. As a generalization of this result, Schoen–Yau [12, p. 18] conjectured in 1985 the nonexistence of proper harmonic maps $\mathbb{D} \to \mathbb{R}^2$. Schoen and Yau related this conjecture with the problem of existence of minimal surfaces in $\mathbb{R}^3$ having hyperbolic conformal type and proper projection into $\mathbb{R}^2$; recall that the coordinate functions of a conformal minimal immersion from a Riemann surface into $\mathbb{R}^3$ are harmonic. In 2001, Forstnerič–Globevnik [7, Theorem 1.4] disproved Schoen–Yau's conjecture. In 2011, Alarcón–Gálvez [1] extended this result to surfaces with finite topology. Although the Schoen–Yau conjecture was solved, its version for minimal surfaces was still open. This problem was settled in the most general and optimum form by Alarcón–López [2, 3], who proved the following result.

Theorem 1.2 (Alarcón–López [2, 3])**.** *Every open Riemann surface $\mathcal{R}$ admits a conformal minimal immersion $X = (X_1, X_2, X_3)\colon \mathcal{R} \to \mathbb{R}^3$, such that $(X_1, X_2)\colon \mathcal{R} \to \mathbb{R}^2$ is a proper map.*

The proof of Alarcón–López is based on a Runge–Mergelyan type theorem for minimal surfaces [2], a powerful tool in the construction of minimal surfaces which has found many applications. Since the coordinate functions of a conformal minimal immersion are harmonic, the full answer to the Schoen–Yau conjecture is immediately derived from Theorem 1.2:

Theorem 1.3. *Every open Riemann surface $\mathcal{R}$ admits a proper harmonic map $\mathcal{R} \to \mathbb{R}^2$.*

An alternative proof of this result was given later by Andrist–Wold [4].

The goal of this article is to give a simple proof of Theorem 1.3. Our proof combines the ideas of Alarcón–López with the classical Runge–Mergleyan Theorem 1.1. Roughly speaking, given an open Riemann surface $\mathcal{R}$, we will construct an expansive sequence of compact sets $\{M_n\}_{n\in\mathbb{N}}$ on $\mathcal{R}$ and harmonic maps $\{h_n : M_n \to \mathbb{R}^2\}_{n\in\mathbb{N}}$ satisfying $h_{n+1} \approx h_n$ on M_n, $n \geq 1$, and $\{h_n(\partial M_n)\}_{n\in\mathbb{N}} \to \infty$. We will ensure that the limit map $h := \lim_{n\to\infty} h_n$ exists and is proper and harmonic.

2. Background

We denote by $\|\cdot\|$ the euclidean norm in $\mathbb{R}^n$. Given a compact topological space K and a continuous function $f : K \to \mathbb{R}^n$, we denote by $\|f\|_K = \max_{a\in K} \|f(a)\|$ the maximum norm of f on K. Given $\zeta \in \mathbb{C}$ we denote by $\Re(\zeta)$ and $\Im(\zeta)$ its real and imaginary parts, respectively.

Let S be a topological surface. We denote by ∂S the topological boundary of S; recall that ∂S is a 1-dimensional topological manifold. Hence, we say that the surface S is *open* if it is not compact and $\partial S = \emptyset$. Given a subset $A \subset S$, we denote by $\overline{A}$ and A° the closure and the interior of A in S, respectively. Given subsets $A, B \subset S$, we write $A \Subset B$ when $\overline{A} \subset B^\circ$. A subset $A \subset S \setminus \partial S$ is called a *bordered region* in S if A is a compact topological surface with the topology induced by S and $\partial A \neq \emptyset$; in particular, ∂A consists of a finite family of pairwise disjoint Jordan curves. If S is a differentiable surface, a bordered region A on S is called *differentiable* if ∂A is differentiable.

Let X and Y be two topological spaces. A map $f : X \to Y$ is called *proper* if $f^{-1}(C)$ is a compact subset of X for any compact subset $C \subset Y$. If f is continuous and Y is Hausdorff, then f is proper if and only if for any divergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in X (i.e., leaving any compact set), the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ is divergent in Y.

Recall that a *Riemann surface* (without boundary) $\mathcal{R}$ is a 1-dimensional complex manifold and every open set of a Riemann surface is canonically a Riemann surface by restriction of charts.

From now on, $\mathcal{R}$ will denote an open Riemann surface.

A function $\phi : \mathcal{R} \to \mathbb{C}$ is called *holomorphic* if the composition with any chart of $\mathcal{R}$ is a holomorphic function; equivalently, if for any point $p \in \mathcal{R}$ there exists a chart around $p \in \mathcal{R}$ such that the composition with ϕ is again a holomorphic function.

Definition 2.1. Let $\mathcal{R}$ be an open Riemann surface. A function $h : \mathcal{R} \to \mathbb{R}$ is called *harmonic* if its composition with any chart is harmonic; equivalently, if for any point $p \in \mathcal{R}$ there exists a chart around $p \in \mathcal{R}$ such that the composition is harmonic. A map $(h_1, \dots, h_n) : \mathcal{R} \to \mathbb{R}^n$, $n \in \mathbb{N}$, is called *harmonic* if $h_j : \mathcal{R} \to \mathbb{R}$ is harmonic for all $j = 1, \dots, n$.

Recall that, since the changes of charts in a Riemann surface are biholomorphisms and the composition of a harmonic function with a biholomorphism is again harmonic, the notion of harmonicity is well-defined on a Riemann surface. Furthermore, a function $h : \mathcal{R} \to \mathbb{R}$ is harmonic if and only if for any simply connected open set $D \subset \mathcal{R}$ there exists a holomorphic function $\phi : D \to \mathbb{C}$ such that $h|_D = \Re(\phi)$.

A compact subset $K \subset \mathcal{R}$ is called *Runge* if $\mathcal{R} \setminus K$ has no relatively compact connected components in $\mathcal{R}$.

Theorem 2.2 (Runge–Mergelyan)**.** *Let $\mathcal{R}$ be an open Riemann surface and let $K \subset \mathcal{R}$ be a compact Runge subset. Given a continuous function $f : K \to \mathbb{C}$ which is holomorphic on K° and given $\epsilon > 0$, there exists a holomorphic function $F : \mathcal{R} \to \mathbb{C}$ such that $\|F - f\|_K < \epsilon$.*

3. Proof of Theorem 1.3

Theorem 1.3 is a consequence of the following more general result, concerning the existence of holomorphic functions into $\mathbb{C}^2$.

Theorem 3.1. *Let $\mathcal{R}$ be an open Riemann surface. Then there exists a holomorphic function $H = (H_1, H_2) : \mathcal{R} \to \mathbb{C}^2$ such that $\Re(H) = (\Re(H_1), \Re(H_2)) : \mathcal{R} \to \mathbb{R}^2$ is proper.*

This is the main result of the paper; since $\Re(H)$ is harmonic, Theorem 1.3 follows directly. Before going into the proof of Theorem 3.1 we need some preparations.

Lemma 3.2. *For any open Riemann surface $\mathcal{R}$ there exists a sequence of bordered regions $\{M_n\}_{n\in\mathbb{N}}$ in $\mathcal{R}$ such that*

(i) M_n is a differentiable bordered region and it is Runge and connected for all $n \in \mathbb{N}$;

(ii) $\{M_n\}_{n\in\mathbb{N}}$ is an exhaustive sequence, that is, $M_n \Subset M_{n+1}$ $\forall n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} M_n = \mathcal{R}$;

(iii) $\chi(M_{n+1} \setminus M_n^\circ) \in \{-1, 0\}$ $\forall n \in \mathbb{N}$, where $\chi(M)$ denotes the Euler characteristic of the region M.

Proof. Let $\{U_n\}_{n\in\mathbb{N}}$ be a exhaustive sequence of $\mathcal{R}$ by connected (differentiable) bordered regions; such sequences trivially exists. Let us first show that we can find a new exhaustion $\{V_n\}_{n\in\mathbb{N}}$ of $\mathcal{R}$ by connected Runge regions. Indeed, if U_1 is Runge we define $V_1 = U_1$; otherwise, we define V_1 as the union of U_1 with all the bounded connected components of $\mathcal{R} \setminus U_1$. Therefore V_1 is Runge and connected. Inductively, for any $n \geq 2$ let V_n be the union of V_{n-1}, U_n and all the bounded connected components of $\mathcal{R} \setminus U_n$. This implies that V_n is Runge and connected. As $U_n \subset V_n$ and $V_n \Subset V_{n+1}$ $\forall n \in \mathbb{N}$, the sequence $\{V_n\}_{n\in\mathbb{N}}$ is an exhaustion of $\mathcal{R}$ by Runge connected bordered regions.

The properties (i) and (ii) of the lemma are formally satisfied by $\{V_n\}_{n\in\mathbb{N}}$. The second step of the proof consists of adding convenient terms to the exhaustion $\{V_n\}_{n\in\mathbb{N}}$ in order to guarantee property (iii).

We consider now two consecutive regions V_m and V_{m+1}, $m \in \mathbb{N}$. Set $A := V_m$, $B := V_{m+1}$ and recall that $A \Subset B$. Let $n := -\chi(B \setminus A^\circ)$.

Claim 3.3. There exist compact sets $N_1, \dots, N_{n-1}$ in $\mathcal{R}$ such that

- $A \Subset N_1 \Subset N_2 \Subset \cdots \Subset N_{n-1} \Subset B$;
- $\chi(N_1 \setminus A^\circ)$, $\chi(N_i \setminus N_{i-1}^\circ)$ and $\chi(B \setminus N_{n-1}^\circ)$ take values in $\{-1, 0\}$, for $i = 2, \dots, n-1$.

Proof. We proceed by induction on n. If $n \in \{-1, 0\}$ there is nothing to prove. Suppose the claim is true when $-\chi(B \setminus A^\circ) \leq n$, $n \in \mathbb{N}$, and let us prove it in the case $-\chi(B \setminus A^\circ) = n + 1$. Recall that A and B are connected Runge regions and $A \Subset B$. Hence, (I) A and $B \setminus A^\circ$ have at least one common boundary component γ_1, thus satisfying $\gamma_1 \subseteq \partial A \cap \partial(B \setminus A^\circ)$; and (II) $B \setminus A^\circ$ has at least one boundary component γ_2 which does not intersect A (in particular, $\gamma_1 \neq \gamma_2$ and so, $\partial(B \setminus A^\circ)$ is not connected).

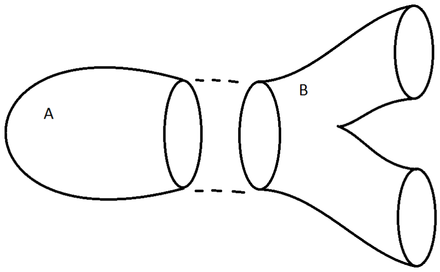

Figure 1: Possibility 1: adding a boundary component.

Let us call g the genus of $B \setminus A^\circ$, and $k \geq 2$ the number of connected components of $\partial(B \setminus A^\circ)$. It follows that $\chi(B \setminus A^\circ) = 2 - 2g - k$. Since $-\chi(B \setminus A^\circ) = n + 1 > 1$ (that is, $2g + k \geq 4$), properties (I) and (II) ensure the existence of a compact region W in $\mathcal{R}$ such that:

(i) W has genus 0 and three boundary components;

(ii) $W \subset B$, $\gamma_2 \subseteq \partial W$ and $W \cap A = \emptyset$;

(iii) if $\gamma \subset \partial W$ is a boundary component of W, then either $\gamma \subset \partial B$ or $\gamma \subset B^\circ$.

Property (iii) is equivalent to the fact that $\partial W \cap \partial B$ has either one or two connected components.

Finally, we define $B_* := \overline{B \setminus W}$. Now we observe that B_* is Runge and connected and also, $A \Subset B_* \Subset B$, $\chi(B_* \setminus A^\circ) = -n$ and $\chi(B \setminus B_*^\circ) = -1$. By the induction hypothesis applied to the pair $A \Subset B_*$, there exist connected Runge compact sets $N_1, \ldots, N_{n-2}$ in $\mathcal{R}$ such that $A \Subset N_1 \Subset \cdots \Subset N_{n-2} \Subset B_*$, $\chi(N_1^\circ \setminus A) \in \{-1, 0\}$, $\chi(B_*^\circ \setminus N_{n-2}) \in \{-1, 0\}$, and $\chi(N_i^\circ \setminus N_{i-1}) \in \{-1, 0\}$ for $i = 2, \ldots, n-2$. Setting $N_{n-1} := B_*$, the sequence of connected Runge compact sets $N_1, \ldots, N_{n-1}$ proves the inductive step and concludes the proof of the claim. □

The sequence $\{M_n\}_{n\in\mathbb{N}}$ that satisfies the statement of the lemma is generated by the process described in the Claim 3.3 applied to each pair $V_m \Subset V_{m+1}$ with $-\chi(V_{m+1} \setminus V_m^\circ) > 1$. We only have to add the new necessary terms and re-enumerate the arising sequence accordingly. □

Remark 3.4. It is interesting to think on the topological operations used in Lemma 3.2. The way we change to the next term is with an Euler characteristic change of value -1 or 0. We can study both in detail:

- Case $\chi(B \setminus A^\circ) = 0$. The compact set B has the same genus and the same number of boundary components than A, hence $B^\circ \setminus A$ is a finite union of pairwise disjoint annuli.
- Case $\chi(B \setminus A^\circ) = -1$. We put $W := B \setminus A^\circ$ and recall that W must have at least two boundary components ($k \geq 2$), one of them contained in ∂A and the other ones disjoint to A and contained in ∂B. Since $\chi(W) = 2 - 2g - k = -1$, where g is the genus of W, we have that $g = 0$ and $k = 3$. This case is possible only in two different topological situations, illustrated in Figures 1 and 2.

We continue with the following lemma whose proof is based on the Runge–Mergelyan Theorem.

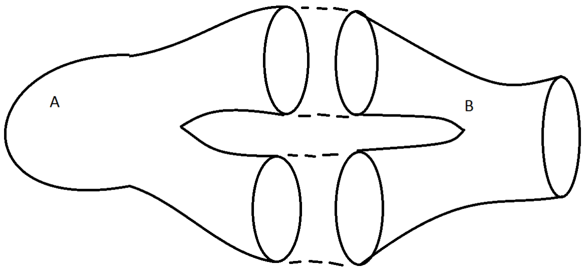

Figure 2: Possibility 2: adding a handle and removing a boundary component.

Lemma 3.5. *Let $\mathcal{R}$ be an open Riemann surface. Let A and B be bordered regions of $\mathcal{R}$ such that $A \Subset B$ and $\chi(B \setminus A^\circ) \in \{-1, 0\}$. Let $\tau > 0$ be a positive number and let $f = (f^1, f^2)\colon A \to \mathbb{C}^2$ be a continuous function, which is holomorphic on A°, and such that $\max\{\Re(f^1), \Re(f^2)\} > \tau$ on ∂A. Then, for any $\delta > 0$, there exists a continuous function $F\colon B \to \mathbb{C}^2$, which is holomorphic on B°, and satisfying the following properties:*

(a) $\|F - f\|_A < \delta$;

(b) $\max\{\Re(F^1), \Re(F^2)\} > \tau$ *on* $B \setminus A$;

(c) $\max\{\Re(F^1), \Re(F^2)\} > \tau + 1$ *on* ∂B.

Proof. We distinguish cases depending on the value of the Euler characteristic of $B \setminus A^\circ$.

Case 1: $\chi(B \setminus A^\circ) = 0$.

In this case $B \setminus A^\circ = A_1 \cup \cdots \cup A_m$, where each A_i is an annulus for all $i \in \{1, \dots, m\}$. In order to simplify notation we will suppose that $m = 1$. The general case is almost identical and consists of applying the same argument to each annulus A_i. Therefore, $B \setminus A^\circ$ is an annulus.

So, $\partial(B \setminus A^\circ)$ consists of two disjoint connected components, that is, $\partial(B \setminus A^\circ) = c \cup d$ where $c = \partial A$ and $d = \partial B$. Since $\max\{\Re(f^1), \Re(f^2)\} > \tau$ on ∂A, we can find an open cover $\widetilde{\Sigma} = \{O_\lambda : \lambda \in \Lambda\}$ of c such that

$$\Re(f^1) > \tau \quad \text{or} \quad \Re(f^2) > \tau, \text{ on each } O_\lambda,\ \lambda \in \Lambda. \tag{1}$$

Since ∂A is compact, there exists a finite subcover $\Sigma = \{O_1, \dots, O_k\}$ of ∂A contained in $\widetilde{\Sigma}$. Take arcs $\alpha_1, \dots, \alpha_n$ in c such that the following properties are satisfied:

- $\alpha_j \subset O_{h(j)}$ for some $h(j) \in \{1, \dots, k\}$, $\forall j = 1 \dots, n$;
- $\cup_{j=1}^n \alpha_j = c$;
- $\alpha_{j_1}^\circ \cap \alpha_{j_2}^\circ = \emptyset$, $\forall j_1, j_2 \in \{1, \dots, n\}$, $j_1 \neq j_2$.

We denote by $p_j \in c$ the initial point of the curve α_j, $\forall j = 1, \dots, n$. We relabel the arcs α_j, $j = 1, \dots, n$, in order to ensure that the final point of α_{j-1} is the initial point p_j of α_j, for any $j > 1$. We adopt the cyclic notation, $p_1 = p_{n+1}$, to identify the initial point of α_1 and the final point of α_n.

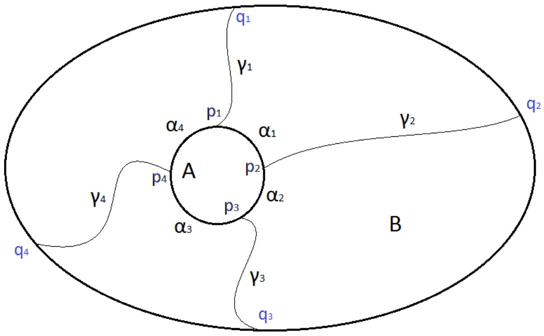

Figure 3: $B \setminus A^\circ$.

Let (I_1, I_2) be subsets of $\{1, \dots, n\}$ satisfying: (1) $I_1 \cup I_2 = \{1, \dots, n\}$ and $I_1 \cap I_2 = \emptyset$; and (2) if $j \in I_\mu$ then $\Re(f^\mu) > \tau$ on α_j, $\mu \in \{1, 2\}$. We consider now a family of non-intersecting simple curves from c to d that we denote $\{\gamma_j\}_{j=1,\dots,n}$. We suppose that the initial point of γ_j is $p_j \in c$ and we call its final point $q_j \in d$. It follows that $\gamma_j \cap (c \cup d) = \{p_j, q_j\}$ for any $j = 1, \dots, n$; see Figure 3.

We define now an auxiliary continuous function $g = (g^1, g^2) \colon A \cup \gamma_1 \cup \cdots \cup \gamma_n \to \mathbb{C}^2$, which is holomorphic on A and satisfies the following properties:

(i) $g|_A = f$;

(ii) if $j - 1 \in I_\mu$ then $\Re(g^\mu) > \tau$ on γ_j and $\Re(g^\mu(q_j)) > \tau + 1$ $\forall j = 1, \dots, n$ (here, we call $\alpha_0 = \alpha_n$ and $q_0 = q_n$);

(iii) if $j \in I_\mu$ then $\Re(g^\mu) > \tau$ on γ_j and $\Re(g^\mu(q_j)) > \tau + 1$ $\forall j = 1, \dots, n$;

where $\mu \in \{1, 2\}$. Such a function g exists due to properties (1) and (2) above.

Since A is Runge, the set $M = A \cup \gamma_1 \cup \cdots \cup \gamma_n$ is also Runge and the Runge–Melgerlyan Theorem gives a continuous function $G \colon B \to \mathbb{C}^2$, which is holomorphic on B° and satisfies the following properties:

(iv) $\|G - g\|_A < \delta/2$ on A; here, δ is the positive number given in the statement of the lemma;

(v) $\max\{\Re(G^1), \Re(G^2)\} > \tau$ on $\partial A = c$;

(vi) if $j - 1 \in I_\mu$ for $\mu \in \{1, 2\}$, then $\Re(G^\mu) > \tau$ on γ_j and $\Re(G^\mu(q_j)) > \tau + 1$, $\forall j = 1, \dots, n$;

(vii) if $j \in I_\mu$ for $\mu \in \{1, 2\}$, then $\Re(G^\mu) > \tau$ on γ_j and $\Re(G^\mu(q_j)) > \tau + 1$, $\forall j = 1, \dots, n$.

Summarizing, the function G formally satisfies properties (a), (b), and (c) on the set M and, by continuity, in a neighbourhood of M, but not necessary in the whole B.

Given $j \in \{1, \dots, n\}$, there is an open neighbourhood Γ_j on B of $\gamma_j \cup \alpha_j \cup \gamma_{j+1}$ such that G still satisfies (a), (b), and (c) in any set Γ_j, $j \in \{1, \dots, n\}$. More concretely, if $j \in I_\mu$ for $\mu \in \{1, 2\}$ then

$$\Re(G^\mu) > \tau \text{ on } \Gamma_j \qquad \text{and} \qquad \Re(G^\mu) > \tau + 1 \text{ on } \Gamma_j \cap \partial B, \tag{2}$$

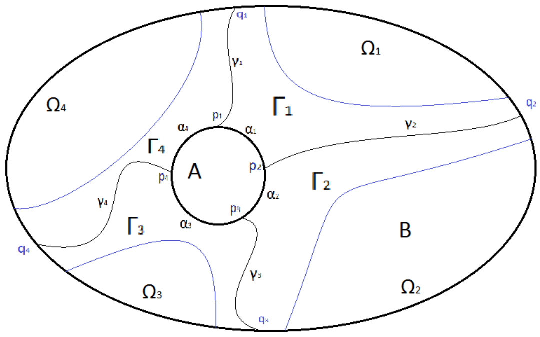

Figure 4: $B \setminus A^\circ$.

$\forall j = 1, \dots, n$. We introduce some more notation. For any fixed $j \in \{1, \dots, n\}$, we consider the topological closed disk in $B \setminus A^\circ$ whose boundary contains the set $\gamma_j \cup \alpha_j \cup \gamma_{j+1}$ and is disjoint from α_l, $l \neq j$. We call Ω_j the complement of Γ_j in this disk; see Figure 4.

Set:

(viii) $\Omega^1 = \bigcup_{j \in I_1} \Omega_j$;

(ix) $\Omega^2 = \bigcup_{j \in I_2} \Omega_j$;

(x) $\Gamma^1 = \bigcup_{j \in I_1} \Gamma_j$;

(xi) $\Gamma^2 = \bigcup_{j \in I_2} \Gamma_j$.

It follows that

$$B = A \cup \Gamma^1 \cup \Gamma^2 \cup \Omega^1 \cup \Omega^2. \tag{3}$$

Consider the functions

$$\widetilde{G}^1 \colon A \cup \Gamma^1 \cup \Omega^2 \to \mathbb{C}, \quad \widetilde{G}^1 = \begin{cases} G^1 & \text{in } A \cup \Gamma^1, \\ \tau + 2 & \text{in } \Omega^2, \end{cases} \tag{4}$$

$$\widetilde{G}^2 \colon A \cup \Gamma^2 \cup \Omega^1 \to \mathbb{C}, \quad \widetilde{G}^2 = \begin{cases} G^2 & \text{in } A \cup \Gamma^2, \\ \tau + 2 & \text{in } \Omega^1, \end{cases} \tag{5}$$

and recall that $(A \cup \Gamma^1) \cap \Omega^2 = \emptyset$ and $(A \cup \Gamma^2) \cap \Omega^1 = \emptyset$.

The sets $A \cup \Gamma^1 \cup \Omega^2$ and $A \cup \Gamma^2 \cup \Omega^1$ are Runge in B, whereas the functions $\widetilde{G}^1$ and $\widetilde{G}^2$ are continuous, and holomorphic on the interior. Hence, by the Runge Theorem, there exist two holomorphic functions on B, which we call F^1 and F^2, that approach $\widetilde{G}^1$ and $\widetilde{G}^2$, respectively. Then, if the approximation is close enough, $F = (F^1, F^2) \colon B \to \mathbb{C}^2$ is holomorphic and satisfies:

(xii) $\|F - \widetilde{G}\|_A < \delta/2$;

(xiii) F^1 approaches $\widetilde{G^1}$ in $A \cup \Gamma^1 \cup \Omega^2$;

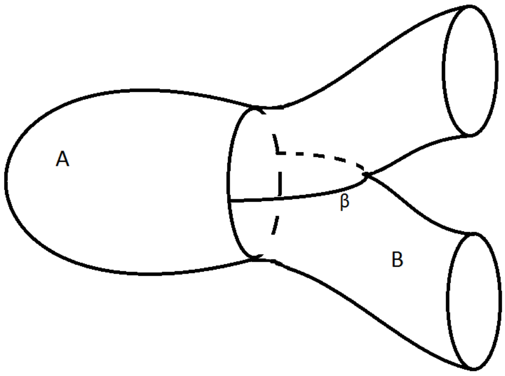

Figure 5: The arc β.

(xiv) F^2 approaches $\widetilde{G^2}$ in $A \cup \Gamma^2 \cup \Omega^1$.

Let us check that F solves the Lemma. Indeed:

- By properties (iv) and (v), $\tilde{G} = G$ on A, and so (xii) gives $\|F - G\|_A < \delta/2$. Thus, taking into account (i) and (iv), we get $\|F - f\|_A < \|F - G\|_A + \|G - f\|_A < \delta$.
- In Γ^1, we have $\Re(G^1) > \tau$ (by equation (2) and property (x)) and so, $\Re(F^1) > \tau$ on Γ^1 provided the approximation in (xiii) is close enough; take into account equation (4). Hence, $\max\{\Re(F^1), \Re(F^2)\} > \tau$ on Γ^1.
- In Γ^2, we have $\Re(G^2) > \tau$ (by equation (2) and property (xi)) and so, $\Re(F^2) > \tau$ on Γ^2 provided the approximation in (xiv) is close enough; use equation (5). Therefore, $\max\{\Re(F^1), \Re(F^2)\} > \tau$ on Γ_2.
- In Ω^1, we have $\Re(\widetilde{G^2}) > \tau$ (by equation (2) and property (viii)) and so, $\Re(F^2) > \tau$ on Ω^1 provided the approximation in (xiv) is close enough. Thus $\max\{\Re(F^1), \Re(F^2)\} > \tau$ on Ω^1.
- In Ω^2, we have $\Re(\widetilde{G^1}) > \tau$ (by equation (2) and property (ix)) and so, $\Re(F^1) > \tau$ on Ω^2 provided the approximation in (xiii) is close enough. Hence $\max\{\Re(F^1), \Re(F^2)\} > \tau$ on Ω^2.

We finish the discussion with the set ∂B. On the one hand, we have $\Re(\widetilde{G^2}) > \tau + 1$ on Ω^1 (by equations (2) and (5)). On the other hand, $\Re(\widetilde{G^1}) > \tau + 1$ on Ω^2 (by equations (2) and (4)). Finally, $\max\{\Re(G^1), \Re(G^2)\} > \tau + 1$ on $\partial B \setminus (\Omega^1 \cup \Omega^2) = (\Gamma^1 \cup \Gamma^2) \cap \partial B$. Indeed, $G^1 > \tau + 1$ on $\partial B \cap \Gamma^1$ and $G^2 > \tau + 1$ on $\partial B \cap \Gamma^2$; see equation (2). Thus, $\max\{\Re(F^1), \Re(F^2)\} > \tau + 1$ in ∂B.

This concludes the proof in case 1.

Case 2: $\chi(B \setminus A^\circ) = -1$.

By Remark 3.4, B can be described as a neighbourhood of the set that we obtain when we add an arc in $B \setminus A$ to A with initial point and final point in ∂A. We call this arc β and we observe that $A \cup \beta$ is a deformation retract of B; see Figures 5 and 6.

Consider a continuous function $g\colon A \cup \beta \to \mathbb{C}^2$, which is holomorphic on A° and satisfies $g = f$ on A and $\max\{\Re(g^1), \Re(g^2)\} > \tau$ on β. By Runge's Theorem we may approximate g on $A \cup \beta$ by holomorphic

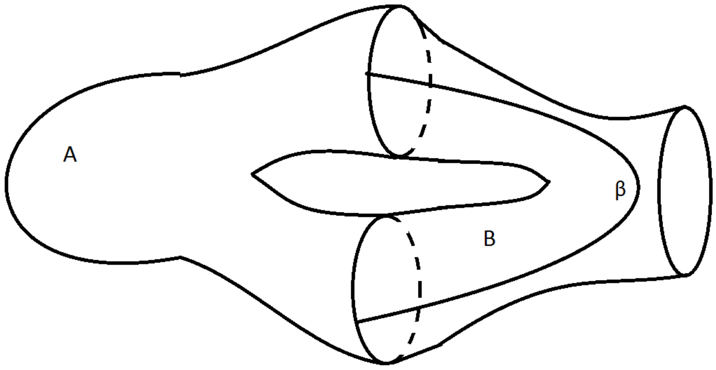

Figure 6: The arc β.

functions $\hat{f}$ on B. If we take a closed neighbourhood $\hat{A}$ of $A \cup \beta$ on B° with $\chi(B \setminus \hat{A}^\circ) = 0$ (recall that $A \cup \beta$ is a deformation retract of B), and the approximation is close enough, the function $\hat{f}|_{\hat{A}}$ formally satisfies the hypothesis (a), (b), and (c) of the lemma. This reduces the proof to Case 1. $\square$

Proof of Theorem 3.1. Let $\{M_n\}_{n\in\mathbb{N}}$ be an exhaustive sequence of Runge and connected bordered regions in $\mathcal{R}$ such that $\chi(M_{n+1} \setminus M_n) \in \{-1, 0\}$ for all $n \in \mathbb{N}$. Such sequences exists by Lemma 3.2. Given a sequence of real numbers $\{\epsilon_n\}_{n\in\mathbb{N}}$, $\epsilon_n > 0$, a recursive use of Lemma 3.5 supplies a sequence of continuous functions $f_n = (f_n^1, f_n^2)\colon M_n \to \mathbb{C}^2$, $n \in \mathbb{N}$, satisfying:

(a) f_n is holomorphic on M_n°, $\forall n \in \mathbb{N}$;

(b) $\|f_{n+1} - f_n\|_{M_n} < \epsilon_n$, $\forall n \in \mathbb{N}$;

(c) $\max\{\Re(f_n^1), \Re(f_n^2)\} > n$ on ∂M_n, $\forall n \in \mathbb{N}$;

(d) $\max\{\Re(f_{n+1}^1), \Re(f_{n+1}^2)\} > n$ on $M_{n+1} \setminus M_n^\circ$, $\forall n \in \mathbb{N}$.

Indeed, for the basis of the induction, choose any continuous function $f_1\colon M_1 \to \mathbb{C}^2$, which is holomorphic on M_1°, and satisfies $\max\{\Re(f_1^1), \Re(f_1^2)\} > 1$ on ∂M_1. For instance, we may take f_1 to be a suitable constant in $\mathbb{C}^2$. For the inductive step, let $n \in \mathbb{N}$ and suppose that we have functions $f_1\colon M_1 \to \mathbb{C}^2, \dots, f_n\colon M_n \to \mathbb{C}^2$ satisfying formally the above properties. Since $M_n \Subset M_{n+1}$ and $\chi(M_{n+1}^\circ \setminus M_n) \in \{-1, 0\}$, Lemma 3.5 applied to $\tau = n$ and $\delta = \epsilon_n$ gives a continuous function $F = (F^1, F^2)\colon M_{n+1} \to \mathbb{C}^2$, which is holomorphic in M_{n+1}°, and satisfies $\|F - f_n\| < \epsilon_n$. In addition, $\max\{\Re(F^1), \Re(F^2)\} > n$ on $M_{n+1} \setminus M_n$ and $\max\{\Re(F^1), \Re(F^2)\} > n+1$ on ∂M_{n+1}. Obviously, we finish the induction setting $f_{n+1} = F$.

Let $\{f_n\colon M_n \to \mathbb{C}^2\}_{n\in\mathbb{N}}$ be the sequence we have already found satisfying (a)–(d). Let us see first that, up to a suitable choice of the numbers ϵ_n, the sequence f_n converges uniformly on compact sets of $\mathcal{R}$. It is enough to prove that (for a good choice of the ϵ_n) given $\epsilon > 0$ and a compact set $K \subset \mathcal{R}$, there exists $n_0 \in \mathbb{N}$ such that if $p, q \geq n_0$ then $\|f_p - f_q\|_K \leq \epsilon$. It is required that n_0 is large enough to satisfy that $K \subset M_{n_0}$, so that f_p and f_q are well defined on K. Indeed, if we take the sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ such that

$\sum_{n=1}^{\infty} \epsilon_n < +\infty$, we can consider n_0 such that $\sum_{n=n_0+1}^{\infty} \epsilon_n < \epsilon$ and $K \subset M_{n_0}$. Then, given $p, q \geq n_0$, $p > q$,

$$\|f_p - f_q\|_K = \| \sum_{k=1}^{p-q} (f_{q+k} - f_{q+k-1}) \|_K \leq \sum_{k=1}^{p-q} \|f_{q+k} - f_{q+k-1}\|_{M_{q+k-1}} < \sum_{k=1}^{p-q} \epsilon_{q+k-1} = \sum_{n=q}^{p} \epsilon_n < \epsilon.$$

Therefore, $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with the maximum norm and, consequently, it converges uniformly on compact sets to a function $f\colon \mathcal{R} \to \mathbb{C}^2$. Furthermore, the convergence Harnack theorem asserts that f is holomorphic and so, $\Re(f)\colon \mathcal{R} \to \mathbb{R}^2$ is harmonic. On the other hand, if we fix ϵ and n_0 as above and we take limits in the previous estimation, we obtain that

$$\|f - f_n\|_{M_n} \leq \epsilon, \quad \forall n \geq n_0. \tag{6}$$

To finish the proof, let us check that $\Re(f)\colon \mathcal{R} \to \mathbb{R}^2$ is proper. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{R}$ be a divergent sequence. Then for each $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that $x_n \in M_{m_n} \setminus M_{m_n-1}$ and, by (d), $\|\Re(f_n(x_n))\| > m_n$. Hence, using (6), we deduce that $\|\Re(f(x_n))\| > m_n - \epsilon$. But now, as $\{x_n\}_{n\in\mathbb{N}}$ is divergent on $\mathcal{R}$ and $\{M_n\}_{n\in\mathbb{N}}$ is increasing, we have that m_n depends on n in such a way that if $n \to +\infty$, then $m_n \to +\infty$. Therefore, it is clear that $\|\Re(f(x_n))\|_{M_n} \to +\infty$ as $n \to +\infty$, and $\{\Re(f(x_n))\}$ is a divergent sequence. Thus, $\varphi = \Re(f)\colon \mathcal{R} \to \mathbb{R}^2$ is proper, which concludes the proof. □

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